Int. J. Solids Structures Vol. 25, No. 7, pp. 715–728, 1989
Printed in Great Britain.

0020-7683/89 $3.00 + .00

INTERACTIVE BUCKLING IN THIN-WALLED BEAMS—I. THEORY

H. MOLLMANN and P. GOLTERMANN†
Department of Structural Engineering, Technical University of Denmark, 2800 Lyngby, Denmark

(*Received* 29 *April* 1987; *in revised form* 21 *October* 1988)

Abstract—A method is derived for the analysis of mode interaction in thin-walled elastic beams. A nonlinear plate theory is employed for the plate segments of the beam, in which the exact nonlinear expressions are used for the middle surface strain measures, but the bending measures are linearized. The beam is subjected to a combination of axial compression and a constant bending moment, and it is assumed to be simply supported at the ends. In the calculation of the total potential energy, the influence of the prebuckling deformations is neglected. The finite strip method is used with the transverse variation of all three displacement components described by cubic polynomials in the arc length. The nonlinear mode interaction is analysed by means of Koiter's asymptotic theory of stability. Some applications of the method to representative problems are presented in a subsequent paper by the authors. This shows that significant mode interaction and imperfection sensitivity occur in these structures.

NOTATION

a_{ijk}, a_{ijkl}	3-index and 4-index coefficients in nonlinear equilibrium equations, defined by eqn (16)
a_{ijI}	3-index coefficient with one buckling mode replaced by one of the remaining eigenfunctions $\mathbf{u}_I$
b	width of finite strip
b_{ijk}	auxiliary 3-index coefficient, eqn (27)
b_{ijkl}, c_{ijkl}	auxiliary 4-index coefficients, eqn (29a)
c_{iN}^m	coefficients in Fourier expansions of approximate displacements, eqn (17c)
E	Young's modulus
$E_{\alpha\beta\lambda\mu}$	elasticity tensor, eqn (5b)
$f_N(y)$	functions describing transverse variation of approximate displacements, eqn (17b)
$\mathcal{F}_{11}$	auxiliary energy functional, eqn (16b)
I_2, I_3	principal moments of inertia of cross-section
$k^m, \bar{k}^m$	contributions to stiffness matrix of a strip, associated with the mth harmonic, eqn (20a)
$\underline{K}^m, \underline{\bar{K}}^m$	contributions to stiffness matrix of complete structure, associated with the mth harmonic, eqn (21)
L	length of beam
$M^{\alpha\beta}$	internal moments in plate
m_i	wave number associated with the ith buckling mode
$N^{\alpha\beta}$	internal forces in plate
N_b	number of buckling modes
$\mathbf{r}_{ij}^m, \mathbf{R}_{ij}^m$	auxiliary functionals, eqns (24) and (25)
t	plate thickness
u_i	displacement components
$\mathbf{u}$	displacement vector with components u_i
$\mathbf{u}_i$	displacement vector of the ith buckling mode
$\mathbf{u}_{ij}, \mathbf{u}_{ijk}$	2nd and 3rd order displacements, eqn (12)
$\mathbf{u}^*$	displacement defining geometric imperfection, eqn (14)
$\bar{u}_i$	components of approximate displacement used in finite strip method, eqn (17a)
$\bar{\mathbf{u}}$	displacement vector with components $\bar{u}_i$
$\mathbf{u}_i^m$	eqn (19)
U_{iN}	eqns (17a) and (17c)
$\mathbf{u}_I$	remaining eigenvector (not a buckling mode), eqn (31)
$\mathbf{v}$	residual displacement, eqn (10)
v_i^m	nodal displacements of strip, associated with mth harmonic, eqn (18)
$\mathbf{v}^m$	vector with components v_i^m
$\mathbf{V}^m$	nodal displacement vector of complete structure, associated with mth harmonic, eqn (21)
$\mathbf{V}_i^m$	nodal displacement vector of the ith buckling mode and associated with the mth harmonic
$\mathbf{V}_{ij}^m$	nodal displacement vector of the 2nd order displacement field and associated with the mth harmonic, eqn (26)
w	displacement component normal to plate

† Also at Ramboll & Hannemann, Consulting Engineers, Teknikerbyen 38, 2830 Virum, Denmark.

x_1, x_2, x_3 local coordinates for plate segment
$\mathbf{x}$ the vector $\{x_1 x_2\}$
x, y alternative notation for x_1, x_2
γ maximum absolute principal strain in plate
$\delta_{\alpha\beta}$ Kronecker's delta
$\varepsilon_{\alpha\beta}$ middle surface strain measures, eqn (1)
η dimensionless coordinate y/b
Φ increment in potential energy in transition from fundamental to adjacent state, eqn (5a)
$\Phi_2, \hat{\Phi}_2, \Phi_3, \Phi_4$ homogeneous functionals making up Φ
$\Phi_{11}, \hat{\Phi}_{11}, \Phi_{111}, \Phi_{1111}$ Fréchet derivatives of functionals Φ_2, $\hat{\Phi}_2$, Φ_3, Φ_4, respectively, see eqns (7b), (13b) and (16c)
$\kappa_{\alpha\beta}$ middle surface bending measures, eqn (2)
λ load factor
λ_i eigenvalue of the ith buckling mode
μ_l Lagrange multipliers, eqns (23a) and (26)
ν Poisson's ratio
ξ_i coefficients of buckling modes, eqn (10)
ξ_i^* coefficients determining geometric imperfections, eqn (14)
ξ_Γ coefficients of remaining eigenvectors, eqn (31)

Indices
Small Latin indices: lower indices i, j, k assume the values 1, 2, 3. An upper index m denotes the wave number.
Small Greek indices $\alpha, \beta, \lambda, \mu$ assume the values 1, 2.
Small script indices i, j, k, l are used to number buckling modes and assume the values $1, 2, \ldots, N_k$.
Capital Latin indices I, J are used to number nodal displacements and assume the values 1 to 12.
Other symbols are defined as they appear in the text.

1. INTRODUCTION

Thin-walled elements are widely used as structural components in many types of metal structures within the fields of civil, mechanical and aeronautical engineering. Because of the slenderness of these structures, their design will often be governed by stability considerations.

Stability failure in a thin-walled structure may occur either as local, plate-type buckling of one or more of the compressed panels, or as overall buckling of the complete structure (flexural column buckling or flexural torsional lateral buckling). In the case of local buckling, the wavelength of the buckling mode is of the order of magnitude of the width of the cross-section, whereas the wavelength of the overall global buckling mode is of the order of the total length of the structure. If the critical loads of these two types of buckling are nearly the same, an interaction between the buckling modes will generally take place, and this may result in a significant reduction of the load-carrying capacity of the structure. Such problems often arise in connection with optimal structural design.

During the last two decades, a considerable amount of theoretical work has been devoted to stability problems involving interaction between local and global buckling modes. These investigations may be divided into two main groups.

The methods in the first group can be characterized in the following manner: the overall behaviour of the structure is described by one-dimensional beam theory, and the interaction is accounted for by the use of a reduced bending stiffness for the beam. This reduced stiffness may be obtained from a theoretical analysis, in which a short segment of the beam is considered and plate theory is applied to the individual panels, or it may be determined empirically from test results. Among the theoretical studies in this group mention should be made of van der Neut's investigations (van der Neut, 1969, 1976), of an idealized column model and of stiffened panels, the paper by Graves-Smith (1969) in which approximate analytical solutions of von Karman's plate equations are utilized, and the papers by Svensson and Croll (1975) and Svensson (1976). The work by Hancock and Bradford also belongs to this group (see e.g. Bradford and Hancock, 1984; Hancock, 1981a, b) and involves a numerical solution of the nonlinear plate equations (including plasticity effects) which is used to determine the reduced bending stiffness. The design rules for thin-walled structures in codes of practice (often formulated in terms of "effective widths" or "effective cross-sections") are usually based on methods of the above-mentioned type.

The methods in the second group are based on the nonlinear theory of elasticity. The thin-walled structure is treated as an assemblage of rectangular plate segments, and an expression for the total potential energy of the structure is derived. The finite strip method or Galerkin's method is often employed, thereby transforming the given continuous system into an equivalent discrete model. The solution may be obtained either by:

(a) iterative solution of the nonlinear equilibrium equations derived from the energy expression of the discrete model, or by
(b) application of Koiter's theory of stability (Koiter, 1945), in which a series of linearized problems are solved successively and used to construct a perturbation solution which decribes the nonlinear behaviour of the structure. Koiter's theory may be applied either to the discrete model or directly to the given continuous system.

Among the investigations belonging to category (a) we may mention a series of papers by Graves-Smith and Sridharan (e.g. Graves-Smith and Sridharan, 1978, 1980a, b; Sridharan and Graves-Smith, 1981), in which the finite strip method is used for the thin-walled structure, and the associated nonlinear equilibrium equations are solved iteratively.

Turning now to category (b) (application of Koiter's theory), one of the early contributions is Tvergaard's investigation of mode interaction in stiffened panels; see Tvergaard (1973). Koiter and his co-workers have studied several problems of mode interaction in thin-walled structures since the early seventies (the van der Neut column and various types of stiffened panels; see e.g. Koiter and Kuiken, 1971; Koiter, 1976; Koiter and Pignataro, 1976). An approximate method is used in which an assumed displacement field involving so-called slowly varying functions is introduced into the framework of Koiter's general theory of stability. Mention should also be made of Byskov's papers on mode interaction in the van der Neut column and in stiffened cylindrical shells (see Byskov and Hutchinson, 1977; Byskov, 1983), which utilize the theory of mode interaction developed in Byskov and Hutchinson (1977).

Finally, a number of recent investigations by Sridharan and his co-workers should be mentioned, see Benito (1983), Benito and Sridharan (1985a, b), Sridharan (1983), Sridharan and Ashraf Ali (1985) and (1986). In these papers, the finite strip method is used in conjunction with Koiter's theory. In the first four papers, the calculations are restricted to a 2-mode analysis involving one local and one global mode, and this will not always provide a sufficiently accurate description of the structural response. However, a 3-mode analysis with two local and one global mode is introduced in the two latest papers, Sridharan and Ashraf Ali (1985, 1986).

Inspection of the papers referred to hitherto will reveal that they all make use of one or more of the following approximations and simplifications.

(1) Some of the nonlinear terms are often neglected in the expressions for the middle surface strain components of the plate segments.

(2) Local buckling is usually described by von Karman plate theory (frequently with the additional assumption that the longitudinal edges between adjacent plate segments do not move), and global buckling is often described by beam theory.

(3) When Koiter's theory is employed, the 2nd order displacements are often approximated by particular solutions to the governing differential equations, thus violating the boundary conditions at the transverse end sections. If the 2nd order displacements are represented by Fourier expansions in the longitudinal direction, only a few terms of these expansions are usually retained in the analysis.

(4) In almost all the above papers, it is assumed that only two modes take part in the interaction (one local and one global mode).

In the present paper, we derive a theory of thin-walled elastic beams capable of describing the whole range of behaviour of these structures from local to global buckling (this is in contrast to most of the existing methods, in which different theories are in fact used to describe local and global buckling). The structure is regarded as an assemblage of plate segments, and the finite strip method in conjunction with Koiter's theory is used for the solution. It will appear from the description of the method in the following sections

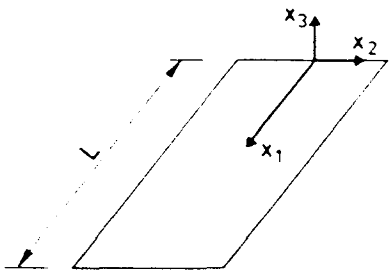

Fig. 1. Plate segment.

that the various approximations and simplifications listed previously are all dispensed with in our proposed theory. A subsequent paper, Goltermann and Møllmann (1989), describes the application of the theory to some representative problems.

The theory was first presented in the second author's thesis, Goltermann (1985), and was used to study several examples of mode interaction in thin-walled beams.

2. GENERAL THEORY OF THIN-WALLED ELASTIC BEAMS

We consider thin-walled prismatic beams composed of plane, rectangular plate segments interconnected along longitudinal edges. Each plate segment is assumed to be of constant thickness which is small compared with the length and width of the segment. For each segment we introduce a local Cartesian coordinate system $x_1x_2x_3$, for which the x_1x_2-plane coincides with the undeformed middle surface, and the x_1-axis is parallel to the longitudinal edges of the undeformed plate, see Fig. 1.

Let $u_i(\mathbf{x})$ denote the components in the x_i-system of the displacement vector $\mathbf{u}$ of the middle surface (measured from the undeformed state). Note that small Latin indices assume the values 1, 2, 3, and that $\mathbf{x} = \{x_1, x_2\}$ denotes the convected coordinates to a generic point on the middle surface. We shall occasionally write $w(\mathbf{x}) = u_3(\mathbf{x})$ for the normal component of the displacement.

Strain and bending measures

The following tensors are introduced to describe the strain and bending of the middle surface of a plate segment.

Strain measures

$$\varepsilon_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha} + u_{i,\alpha}u_{i,\beta}) \tag{1}$$

Bending measures

$$\kappa_{\alpha\beta} = -w_{,\alpha\beta} \tag{2}$$

where small Greek indices assume the values 1, 2, the comma notation is used for partial derivatives, and the summation convention is valid for lower Greek indices and for lower Latin indices.

The use of the complete nonlinear strain expression (1) (including nonlinear terms in all the three displacement components) will enable us to describe overall buckling which involves significant in-plane displacements. This is in contrast to the conventional von Karman plate theory (nonlinear terms only in the normal displacement w) which can merely describe transverse buckling of the plate.

It will be seen that the expression (2) for the bending measures is linearized with respect to the displacements. Let γ denote the maximum absolute principal strain in the plate and assume that the order of magnitude of the displacement gradients is given by $u_{i,\alpha} = O(\gamma^{1/2})$ (moderate rotations). It can then be shown that the relative error of the approximation (2) is of order γ.

Support conditions

The beam is assumed to be supported at the ends by transverse diaphragms which prevent displacements in the planes of the diaphragms but offer negligible resistance to displacements perpendicular to the diaphragms (simple supports). The corresponding geometric boundary conditions are therefore given by

$$u_2 = 0, \quad w = 0 \tag{3}$$

at the ends of the beam ($x_1 = 0$ and $x_1 = L$).

Loading

We assume that the beam is loaded by prescribed axial stresses on the end sections with a plane stress distribution over the cross-section, and that the prescribed stresses are always parallel to the x_1-axis ("dead" loading). The prebuckling equilibrium state (the fundamental state) is then a state of uniaxial longitudinal stress. The loading is specified as the product of a unit loading system and a scalar load factor λ.

Displacements of fundamental state neglected

In the following we shall assume that the displacements from the undeformed to the fundamental state are negligible, and we shall omit these displacements when forming the potential energy expression. This is an excellent approximation in the case of column loading (uniformly distributed prescribed stresses), where the corresponding relative error in the potential energy is of the order of the maximum principal strain of the fundamental state (see Koiter, 1982). However, in the case of moment loading (a non-constant plane distribution of prescribed stresses), the approximation is somewhat less accurate. Thus, in the case of overall lateral buckling of a thin-walled beam subjected to bending about the major principal axis, it can be shown that the omission of the displacements of the fundamental state gives rise to a relative error in the critical moment which may be of noticeable magnitude (of the order I_3/I_2, where I_2 and I_3 ($<I_2$) are the principal moments of inertia of the beam cross-section), although the postbuckling behaviour appears to be only slightly affected by the present approximation (see Goltermann, 1985; Pedersen, 1982).

Omission of the displacements of the fundamental state implies that we ignore the difference between the configurations of the undeformed state and the fundamental state, and we may consequently regard the previously defined displacements u_i as additional displacements from the fundamental state to an adjacent state.

Total potential energy

The omission of the displacements of the fundamental state implies that we have a linear fundamental state with internal forces and moments given by:

$$N^{o}_{11} = \lambda \hat{N}_{11}, \quad N^{o}_{22} = N^{o}_{12} = N^{o}_{21} = 0, \quad M^{o}_{\alpha\beta} = 0 \tag{4}$$

where $\hat{N}_{11}$ is a linear function of x_2 according to the present type of prescribed loading.

Assuming small strains and an elastic material, the increment in potential energy of the beam in the transition from the fundamental state to an adjacent state (at the same load factor λ) is given by

$$\Phi[\mathbf{u}; \lambda] = \int \tfrac{1}{2}\{\lambda \hat{N}_{11} u_{i,1} u_{i,1} + E_{\alpha\beta\lambda\mu}(\varepsilon_{\alpha\beta}\varepsilon_{\lambda\mu} + \tfrac{1}{12} t^2 \kappa_{\alpha\beta}\kappa_{\lambda\mu})\}\, \mathrm{d}A \tag{5a}$$

where the elasticity tensor $E_{\alpha\beta\lambda\mu}$, in the case of a homogeneous and isotropic material, is given by

$$E_{\alpha\beta\lambda\mu} = \frac{Et}{2(1+\nu)}\left(\delta_{\alpha\lambda}\delta_{\beta\mu} + \delta_{\alpha\mu}\delta_{\beta\lambda} + \frac{2\nu}{1-\nu}\delta_{\alpha\beta}\delta_{\lambda\mu}\right). \tag{5b}$$

In this formula, E denotes Young's modulus, ν is Poisson's ratio, t is the thickness of the plate segment, and $\delta_{\alpha\beta}$ is Kronecker's delta. It should be noted that the integration in (5a) is extended over the middle surfaces of *all* the plate segments of the beam, and that the displacement components in each plate segment are referred to the associated local coordinate system. The potential energy (5a) may be written in the form

$$\Phi[\mathbf{u}\,;\lambda] = \Phi_2[\mathbf{u}] + \lambda\dot{\Phi}_2[\mathbf{u}] + \Phi_3[\mathbf{u}] + \Phi_4[\mathbf{u}] \tag{6a}$$

where $\Phi_i[\mathbf{u}]$ and $\dot{\Phi}_i[\mathbf{u}]$ denote homogeneous functionals of the ith degree in the displacement derivatives given by

$$\Phi_2[\mathbf{u}] = \int \tfrac{1}{2}E_{\alpha\beta\lambda\mu}(u_{\alpha,\beta}u_{\lambda,\mu} + \tfrac{1}{12}t^2 w_{,\alpha\beta}w_{,\lambda\mu})\,\mathrm{d}A$$

$$\dot{\Phi}_2[\mathbf{u}] = \int \tfrac{1}{2}\hat{N}_{11}u_{i,1}u_{i,1}\,\mathrm{d}A$$

$$\Phi_3[\mathbf{u}] = \int \tfrac{1}{2}E_{\alpha\beta\lambda\mu}u_{\alpha,\beta}u_{i,\lambda}u_{i,\mu}\,\mathrm{d}A$$

$$\Phi_4[\mathbf{u}] = \int \tfrac{1}{8}E_{\alpha\beta\lambda\mu}u_{i,\alpha}u_{i,\beta}u_{j,\lambda}u_{j,\mu}\,\mathrm{d}A. \tag{6b}$$

3. INTERACTION ANALYSIS FOR CONTINUOUS SYSTEMS

The critical points on the fundamental equilibrium path are determined by the eigenvalues of the linear eigenvalue problem

$$\Phi_{11}[\mathbf{u}, \delta\mathbf{u}] + \lambda\dot{\Phi}_{11}[\mathbf{u}, \delta\mathbf{u}] = 0 \tag{7a}$$

where $\delta\mathbf{u}$ is a kinematically admissible but otherwise arbitrary displacement variation. The bilinear functionals Φ_{11} and $\dot{\Phi}_{11}$ are defined by

$$\Phi_{11}[\mathbf{u}, \mathbf{v}] = \Phi_2''\mathbf{uv}, \quad \dot{\Phi}_{11}[\mathbf{u}, \mathbf{v}] = \dot{\Phi}_2''\mathbf{uv} \tag{7b}$$

where dashes denote Fréchet derivatives.

It can be shown that the problem (7a) in the present case possesses an enumerable set of eigenvalues $\lambda_{\mathscr{I}}$ ($\mathscr{I} = 1, 2, 3, \ldots$) and associated eigenfunctions $u_{\mathscr{I}}(\mathbf{x})$. The eigenvalues are all non-zero, and the eigenfunctions satisfy the orthogonality conditions

$$\Phi_{11}[\mathbf{u}_{\mathscr{I}}, \mathbf{u}_{\mathscr{J}}] = 0, \quad \dot{\Phi}_{11}[\mathbf{u}_{\mathscr{I}}, \mathbf{u}_{\mathscr{J}}] = 0 \quad \text{for} \quad \mathscr{I} \neq \mathscr{J}. \tag{7c}$$

We shall also consider the effect of geometric imperfections. We assume that the configuration of the unloaded imperfect structure is determined by an initial displacement field $\mathbf{u}^*(\mathbf{x})$. Koiter (1945, 1980) has shown that, in the case of small imperfections, the increment in potential energy Φ^* for the imperfect structure is obtained by adding a term $\lambda\dot{\Phi}_{11}[\mathbf{u}^*, \mathbf{u}]$ to the energy expression (6a) for the perfect structure. Hence, for the imperfect structure,

$$\Phi^*[\mathbf{u}\,;\lambda] = \Phi_2[\mathbf{u}] + \lambda\dot{\Phi}_2[\mathbf{u}] + \Phi_3[\mathbf{u}] + \Phi_4[\mathbf{u}] + \lambda\dot{\Phi}_{11}[\mathbf{u}^*, \mathbf{u}]. \tag{8}$$

Suppose that we wish to study the interaction for values of the load factor in an interval

$\lambda_a \leqslant \lambda \leqslant \lambda_b$. We then select a finite number N_b of eigenfunctions (buckling modes) for which the corresponding eigenvalues should include all those contained in the interval $\lambda_a \leqslant \lambda \leqslant \lambda_b$. The following notation is introduced for these quantities:

$$\left.\begin{array}{ll}\text{buckling modes:} & \mathbf{u}_i(\mathbf{x}),\\ \text{associated eigenvalues:} & \lambda_i,\end{array}\right\} \quad i = 1, 2, \ldots, N_b.$$

Note that a small script index denotes the number of a buckling mode, and that the summation convention is valid for script indices.

We now wish to normalize the buckling modes in a suitable manner. The functional $\Phi_2[\mathbf{u}]$ is positive definite (for a properly supported structure) and could therefore be used as our norm, but we shall find it convenient to use the somewhat simpler functional $\dot{\Phi}_2[\mathbf{u}]$ for this purpose [cf. (6b)]. However, here we encounter a minor difficulty. A norm is generally required to be positive definite, but it can be seen from (7a) that if there are both positive and negative eigenvalues (as will occur, e.g. in problems of lateral buckling of beams), then $\dot{\Phi}_2[\mathbf{u}]$ will assume both positive and negative values. In order to deal with this difficulty, we shall always select our buckling modes in such a way that all the associated eigenvalues have the same sign (which may be assumed to be positive). It then follows from (7a) that $\dot{\Phi}_2[\mathbf{u}_i]$ will be negative for all buckling modes, and we shall now normalize the buckling modes in the following manner:

$$-\dot{\Phi}_{11}[\mathbf{u}_{\underline{i}}, \mathbf{u}_{\underline{i}}] = -2\dot{\Phi}_2[\mathbf{u}_i] = 1 \tag{9a}$$

where the repeated underlined indices should not be summed. From (9a) and (7a) it follows that

$$\Phi_{11}[\mathbf{u}_{\underline{i}}, \mathbf{u}_{\underline{i}}] = \lambda_i. \tag{9b}$$

Consider an equilibrium configuration in a neighbourhood of the segment $\lambda_a \leqslant \lambda \leqslant \lambda_b$ of the fundamental equilibrium path. The displacement of this equilibrium configuration is written in the form

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}_i(\mathbf{x})\xi_i + \mathbf{v}(\mathbf{x}), \tag{10}$$

i.e. a linear combination of buckling modes plus a residual displacement $\mathbf{v}$, where $\mathbf{v}$ is required to be orthogonal to the buckling modes in the sense that

$$\dot{\Phi}_{11}[\mathbf{u}_i, \mathbf{v}] = 0. \tag{11}$$

It can be shown (see Mollmann, 1984) that the residual displacement $\mathbf{v}$ can be expressed as a power series of the form

$$\mathbf{v}(\mathbf{x}, \xi_i, \lambda) = \mathbf{u}_{ij}(\mathbf{x}, \lambda)\xi_i\xi_j + \mathbf{u}_{ijk}(\mathbf{x}, \lambda)\xi_i\xi_j\xi_k + \cdots, \tag{12}$$

convergent in the above neighbourhood. The $\mathbf{u}_{ij}$ are called 2nd order displacements, the $\mathbf{u}_{ijk}$ are called 3rd order displacements, etc. In the following, we shall only retain the first term of the expansion (12) (quadratic in the parameters ξ_i).

The 2nd order displacements $\mathbf{u}_{ij}$ are determined by the equation

$$\Phi_{11}[\mathbf{u}_{ij}, \delta\mathbf{v}] + \lambda\dot{\Phi}_{11}[\mathbf{u}_{ij}, \delta\mathbf{v}] + \tfrac{1}{2}\Phi_{111}[\mathbf{u}_i, \mathbf{u}_j, \delta\mathbf{v}] = 0 \tag{13a}$$

where the variation $\delta\mathbf{v}$ is required to be orthogonal to the buckling modes, and

$$\Phi_{111}[\mathbf{u},\mathbf{v},\mathbf{w}] = \phi'''_{3}\mathbf{uvw}. \tag{13b}$$

Assuming that the geometric imperfections are given by

$$\mathbf{u}^*(\mathbf{x}) = \mathbf{u}_i(\mathbf{x})\xi_i^* \tag{14}$$

(primary imperfections), it is found that the equilibrium configurations of the system in the above-mentioned neighbourhood of the fundamental equilibrium path are determined by the following non-linear algebraic equations for the parameters ξ_i (see Byskov and Hutchinson, 1977; Mollmann, 1984).

$$(\lambda_{\underline{i}}-\lambda)\xi_{\underline{i}}+a_{ijk}\xi_j\xi_k+a_{ijkl}\xi_j\xi_k\xi_l = \lambda\xi_i^* \tag{15}$$

where the underlined repeated indices should not be summed. If the complete series expansion (12) had been used, we should have obtained an infinite power series on the left-hand side of (15), and this series has in fact been truncated after the cubic terms in agreement with our truncation in (12).

The coefficients a_{ijk} and a_{ijkl} in (15) are completely symmetric in their indices and are given by

$$\begin{aligned} a_{ijk} &= \tfrac{1}{2}\Phi_{111}[\mathbf{u}_i,\mathbf{u}_j,\mathbf{u}_k] \\ a_{ijkl} &= \tfrac{1}{6}\Phi_{1111}[\mathbf{u}_i,\mathbf{u}_j,\mathbf{u}_k,\mathbf{u}_l] - \tfrac{1}{3}\mathscr{F}_{11}[\mathbf{u}_{ij},\mathbf{u}_{kl};\lambda] \\ &\quad - \tfrac{1}{3}\mathscr{F}_{11}[\mathbf{u}_{ik},\mathbf{u}_{jl};\lambda] - \tfrac{1}{3}\mathscr{F}_{11}[\mathbf{u}_{il},\mathbf{u}_{jk};\lambda] \end{aligned} \tag{16a}$$

where

$$\mathscr{F}_{11}[\mathbf{u},\mathbf{v};\lambda] = \Phi_{11}[\mathbf{u},\mathbf{v}]+\lambda\dot{\Phi}_{11}[\mathbf{u},\mathbf{v}] \tag{16b}$$

and

$$\Phi_{1111}[\mathbf{u},\mathbf{v},\mathbf{w},\mathbf{z}] = \Phi''''_{4}\mathbf{uvwz}. \tag{16c}$$

4. FINITE STRIP METHOD

The plate segments of the beam are now divided into a finite number of longitudinal strips. In each strip, the transverse variation of the three displacement components will be approximated by cubic polynomials, i.e. we introduce approximate displacements $\bar{u}_i$ for a strip given by (see Goltermann, 1985)

$$\bar{u}_i(x,y) = \sum_{N=1}^{4} U_{iN}(x)f_N(y) \tag{17a}$$

where we have changed the notation for the independent variables, i.e.

$$\{x_1x_2\} = \{xy\}.$$

The functions $f_N(y)$ are the following cubic polynomials:

$$\begin{aligned} f_1 &= 2\eta^3-3\eta^2+1, \quad & f_2 &= b(\eta^3-2\eta^2+\eta) \\ f_3 &= -2\eta^3+3\eta^2, \quad & f_4 &= b(\eta^3-\eta^2) \end{aligned} \tag{17b}$$

where $\eta = y/b$, b being the width of the strip.

Further, the functions $U_{iN}(x)$ are taken to be finite trigonometric series as follows:

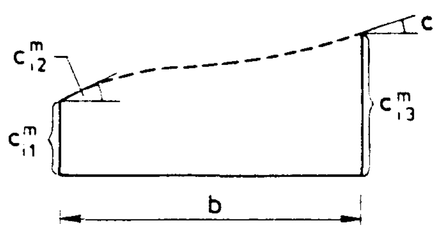

Fig. 2. Geometrical meaning of coefficients c^m_{iN}.

$$U_{1N}(x) = \sum_{m=1}^{M} c^m_{1N} \cos \frac{m\pi}{L} x$$

$$U_{2N}(x) = \sum_{m=1}^{M} c^m_{2N} \sin \frac{m\pi}{L} x$$

$$U_{3N}(x) = \sum_{m=1}^{M} c^m_{3N} \sin \frac{m\pi}{L} x \qquad (17c)$$

where L is the length of the beam and the quantities c^m_{iN} are initially unknown coefficients. Assuming that the origin of the local coordinate system is located at one end of the beam (see Fig. 1), it will be seen that the displacements (17a) satisfy the geometric boundary conditions (3) at the ends of the beam (simple supports).

It follows from the form of the cubic polynomials (17b) and from (17a) and (17c) that the coefficients c^m_{iN} (for fixed values of i and m) are the amplitudes of the mth harmonics in the expansions of the displacement $\bar{u}_i$ and the derivative $\partial \bar{u}_i/\partial y$ at the longitudinal edges of the strip, see Fig. 2.

Most of the previous applications of the finite strip method to the calculation of prismatic plate structures (see e.g. Graves-Smith and Sridharan, 1978; Benito and Sridharan, 1985a, b) have involved the use of linear functions of y to describe the transverse variation of the in-plane displacements u_1 and u_2, but cubic polynomials in y for the normal displacement w. However, a coupling between in-plane and normal displacements of adjacent segments takes place at the longitudinal edges, and it is therefore appropriate to use the same type of functions (cubic polynomials) for all three displacement components, as we have done.

If the approximate displacements (17a) are substituted in the energy expression (8), the potential energy of a strip becomes a 4th degree polynomial in the coefficients c^m_{iN}, and we have in effect replaced the continuous system by an approximate discrete model. This approach was used, e.g. by Graves-Smith and Sridharan (1978, 1980a, b). However, the resulting energy expression gets rather unwieldy, and we prefer to proceed in a different manner. We substitute the displacement approximation (17a) in the governing equations (7a), (13a), (16a) of the continuous system. The resulting algebraic equations for the determination of the modes, the 2nd order displacements, and the coefficients of the nonlinear equations, will in fact coincide with those that are obtained from the energy expression of the discrete model if the perturbation method (Koiter's theory) is used directly for the discrete system. This follows from the fact that the Fréchet derivatives in (7a), (13a), and (16a) for the case of the discrete model reduce to partial derivatives, and we obtain the usual formulae for discrete systems. It will be convenient to introduce a special notation in (17a) for the group of coefficients associated with a particular wave number m in the trigonometric functions in (17c). The 12 coefficients c^m_{iN} associated with wave number m are denoted by

$$\mathbf{v}^m = \{v^m_I\}$$
$$= [c^m_{11} c^m_{12} c^m_{13} c^m_{14} c^m_{21} c^m_{22} c^m_{23} c^m_{24} c^m_{31} c^m_{32} c^m_{33} c^m_{34}]^T. \qquad (18)$$

The v_I^m may be regarded as nodal displacements associated with wave number m, and the displacements (17a) of the strip may then be written in the form

$$\bar{\mathbf{u}}(x,y) = \sum_{m=1}^{M} \mathbf{u}_I^m(x,y)v_I^m \tag{19}$$

where it is understood that the summation convention applies to lower indices but not to upper indices, and $I = 1, 2, 3, \ldots, 12$. In the following, $\Delta\Phi[\mathbf{u};\lambda]$ will denote the contribution of a strip to the total potential energy of the beam. Consider now the quadratic energy contribution $(\Delta\Phi_2[\bar{\mathbf{u}}]+\lambda\Delta\dot{\Phi}_2[\bar{\mathbf{u}}])$ from a strip, where we have inserted the approximate displacement $\bar{\mathbf{u}}(x,y)$ [see (19)]. Each term of the integrands in the quadratic functionals contains a product of two trigonometric functions of x, and because of the orthogonality properties of these functions, all such products involving two different wave numbers vanish when integrated. The result can therefore be written as a sum of contributions, each of which involves only the nodal displacements $\mathbf{v}^m$ associated with one particular wave number, i.e.

$$\Delta\Phi_2[\bar{\mathbf{u}}]+\lambda\Delta\dot{\Phi}_2[\bar{\mathbf{u}}] = \sum_{m=1}^{M} \tfrac{1}{2}(\mathbf{v}^m)^T(\underset{=}{k}^m+\lambda\dot{\underset{=}{k}}^m)\mathbf{v}^m \tag{20a}$$

where $\underset{=}{k}^m$ and $\dot{\underset{=}{k}}^m$ are 12×12 matrices with components given by

$$\begin{aligned} k^m(I,J) &= \Delta\Phi_{11}[\mathbf{u}_I^m,\mathbf{u}_J^m] \\ \dot{k}^m(I,J) &= \Delta\dot{\Phi}_{11}[\mathbf{u}_I^m,\mathbf{u}_J^m]. \end{aligned} \tag{20b}$$

The integrations involved in (20a) are evaluated analytically, both in the x-direction (products of two trigonometric functions of x), and in the y-direction (polynomials in y of up to 6th degree).

At the longitudinal edges of the strip the displacement vector as well as the derivative $\partial\bar{w}/\partial y$ must be continuous when we pass from one strip to the adjacent one. These continuity conditions will be satisfied if we introduce global nodal displacements $\mathbf{V}^m$ which describe the nodal displacements of all the edges, and then express the local nodal displacements $\mathbf{v}^m$ in terms of the global quantities $\mathbf{V}^m$. Summing the contributions from the strips, the quadratic energy functional of the complete beam may then be written in the form

$$\Phi_2[\bar{\mathbf{u}}]+\lambda\dot{\Phi}_2[\bar{u}] = \sum_{m=1}^{M} \tfrac{1}{2}(\mathbf{V}^m)^T(\underset{=}{K}^m+\lambda\dot{\underset{=}{K}}^m)\mathbf{V}^m. \tag{21}$$

It follows [see (7a)] that the modes and eigenvalues of the discrete model are determined by the linear matrix eigenvalue problem

$$(\underset{=}{K}^m+\lambda_r\dot{\underset{=}{K}}^m)\mathbf{V}_r^m = \mathbf{0}. \tag{22a}$$

Note that the x-variation of a mode involves only one wavelength (i.e. $\mathbf{V}_r^m \neq \mathbf{0}$ for only one value of the wave number m). We shall assume that the modes are normalized in the following manner:

$$-\dot{\Phi}_{11}[\bar{\mathbf{u}}_r,\bar{\mathbf{u}}_r] = -(\mathbf{V}_r^m)^T\dot{\underset{=}{K}}^m\mathbf{V}_r^m = 1. \tag{22b}$$

We now select buckling modes $\bar{\mathbf{u}}_r$ for the discrete model ($r = 1, 2, \ldots, N_b$), and we next consider the 2nd order displacements of the discrete model. Inserting the displacement approximation (19) in eqns (13a) for the determination of the 2nd order displacements, and removing the orthogonality restrictions on $\delta\mathbf{v}$ by means of appropriate Lagrange multipliers, we obtain the equations

$$\Phi_{11}[\bar{\mathbf{u}}_{\ell},\delta\bar{\mathbf{u}}]+\lambda\dot{\Phi}_{11}[\bar{\mathbf{u}}_{\ell},\delta\bar{\mathbf{u}}]+\mu_{\ell}\dot{\Phi}_{11}[\bar{\mathbf{u}}_{\ell},\delta\bar{\mathbf{u}}]+\tfrac{1}{2}\Phi_{111}[\bar{\mathbf{u}}_{i},\bar{\mathbf{u}}_{j},\delta\bar{\mathbf{u}}]=0,\quad \dot{\Phi}_{11}[\bar{\mathbf{u}}_{\ell},\bar{\mathbf{u}}_{ij}]=0 \tag{23a}$$

where μ_ℓ ($\ell = 1,2,\ldots,N_b$) are the Lagrange multipliers.

$$\bar{\mathbf{u}}_{ij} = \sum_{m=1}^{M} \mathbf{u}_i^m(x,y)(v_{ij}^m),\quad \delta\bar{\mathbf{u}} = \sum_{m=1}^{M} \mathbf{u}_i^m(x,y)\,\delta v_i^m \tag{23b}$$

and the δv_i^m (and thus $\delta\bar{\mathbf{u}}$) are not subjected to any orthogonality restrictions. The functional $\Phi_{111}[\bar{\mathbf{u}}_i,\bar{\mathbf{u}}_j,\delta\bar{\mathbf{u}}]$ is linear in $\delta\bar{\mathbf{u}}$. The corresponding contribution from a strip is written in the form

$$\Delta\Phi_{111}[\bar{\mathbf{u}}_i,\bar{\mathbf{u}}_j,\delta\bar{\mathbf{u}}] = \sum_{m=1}^{M}\Delta\Phi_{111}[\bar{\mathbf{u}}_i,\bar{\mathbf{u}}_j,\mathbf{u}_i^m]\,\delta v_i^m = \sum_{m=1}^{M}(\mathbf{r}_{ij}^m)^T\delta\mathbf{v}^m \tag{24}$$

which defines the vector $\mathbf{r}_{ij}^m$. Introducing global nodal displacements, and summing the contributions from the strips, we get

$$\tfrac{1}{2}\Phi_{111}[\bar{\mathbf{u}}_i,\bar{\mathbf{u}}_j,\delta\bar{\mathbf{u}}] = \frac{1}{2}\sum_{m=1}^{M}(\mathbf{R}_{ij}^m)^T\delta\mathbf{V}^m. \tag{25}$$

The bilinear functionals Φ_{11} and $\dot{\Phi}_{11}$ in (23a) can be expressed in terms of the matrices $\underline{K}^m$ and $\dot{\underline{K}}^m$ [cf. (20b) and (21)]. From (23a) and (25) we then obtain the M systems of linear equations:

$$\begin{aligned}(\underline{K}^m+\lambda\dot{\underline{K}}^m)\mathbf{V}_{ij}^m+\mu_\ell\dot{\underline{K}}^m\mathbf{V}_\ell^m+\tfrac{1}{2}\mathbf{R}_{ij}^m&=0,\\ (\mathbf{V}_\ell^m)^T\dot{\underline{K}}^m\mathbf{V}_{ij}^m&=0.\end{aligned} \tag{26}$$

This means that we get one system of linear equations for each value of the wave number ($m = 1,2,\ldots,M$) for the determination of the corresponding nodal displacements $\mathbf{V}_{ij}^m$ of the 2nd order displacement field and those Lagrange multipliers μ_ℓ for which the wave number of the associated buckling mode $\bar{\mathbf{u}}_\ell$ is equal to m.

The coefficients in the nonlinear equations are given by (16a). We first consider the coefficients a_{ijk} with three indices. Let us define quantities b_{ijk} as follows:

$$b_{ijk} = \int E_{\alpha\beta\gamma\mu}\bar{u}^i_{\alpha,\beta}\bar{u}^j_{\gamma}\bar{u}^k_{\gamma,\mu}\,\mathrm{d}A. \tag{27}$$

We then find that the three-index coefficients are given by (see Goltermann, 1985)

$$a_{ijk} = \tfrac{1}{2}\Phi_{111}[\bar{\mathbf{u}}_i,\bar{\mathbf{u}}_j,\bar{\mathbf{u}}_k] = \tfrac{1}{2}(b_{ijk}+b_{jki}+b_{kij}). \tag{28}$$

The integrations in the x-direction in (27) are evaluated analytically (products of three trigonometric function of x). The results of these integrations show that $a_{ijk} = 0$ when the sum of the wave numbers associated with the three modes ($m_i+m_j+m_k$) is an even number. The integrations in the y-direction are in each strip computed numerically by means of Gauss' integration formula (the integrands are polynomials of up to 9th degree in y).†

We next consider the coefficients a_{ijkl} with four indices. Let us define quantities b_{ijkl} and c_{ijkl} as follows:

† A similar method is used to evaluate the components of the vector $\mathbf{R}_{ij}^m$, see (25), which also depends on the trilinear functional Φ_{111}.

$$b_{\iota\kappa\mu\nu} = \int E_{\alpha\beta\lambda\mu}\bar{u}'_{\iota,\alpha}\bar{u}'_{\kappa,\beta}\bar{u}'_{\mu,\lambda}\bar{u}'_{\nu,\mu}\,\mathrm{d}A$$

$$c_{\iota\kappa\mu\nu} = \Phi_{11}[\bar{\mathbf{u}}_{\iota\kappa}, \bar{\mathbf{u}}_{\mu\nu}] + \lambda\dot{\Phi}_{11}[\bar{\mathbf{u}}_{\iota\kappa}, \bar{\mathbf{u}}_{\mu\nu}]. \tag{29a}$$

It is then found that the four-index coefficients are given by (see Goltermann, 1985)

$$a_{\iota\kappa\mu\nu} = \tfrac{1}{6}(b_{\iota\kappa\mu\nu} + b_{\iota\mu\kappa\nu} + b_{\iota\nu\kappa\mu}) - \tfrac{2}{3}(c_{\iota\kappa\mu\nu} + c_{\iota\mu\kappa\nu} + c_{\iota\nu\kappa\mu}). \tag{29b}$$

Now the 2nd order displacements of the discrete model are expressed as finite trigonometric series in the x-direction [see (17c)]. When we form the expression $(29a)_2$ for $c_{\iota\kappa\mu\nu}$, each term of the integrands will contain a product of two trigonometric functions of x, the integral of which vanishes if the corresponding two wave numbers differ. The result can therefore be written as a sum of contributions corresponding to each of the wave numbers, and we obtain [cf. also (20a) and (21)]

$$c_{\iota\kappa\mu\nu} = \sum_{m=1}^{M} (\mathbf{V}^m_{\iota\kappa})^T(\underline{K}^m + \lambda\dot{\underline{K}}^m)\mathbf{V}^m_{\mu\nu}. \tag{30}$$

It remains to calculate the quantities $b_{\iota\kappa\mu\nu}$, see $(29a)_1$. If we insert the modes of the discrete model directly into $(29a)_1$, each term of the integrand will contain a product of four trigonometric functions of x, and these products can easily be integrated analytically. However, it is found that the resulting values of $a_{\iota\kappa\mu\nu}$ are rather poorly determined, so that a large number of terms will be required in the trigonometric series in (17c) to attain a sufficient accuracy.

We shall therefore use an alternative method which yields a much improved accuracy. We expand terms of the type $\bar{u}'_{\iota,\alpha}\bar{u}'_{\kappa,\beta}$ in a finite trigonometric series in the x-direction, using the same trigonometric functions as those appearing in the expansion of the derivatives $\bar{u}'_{\iota\kappa,\beta}$ of the components of the 2nd order displacements. The resulting expansions are now substituted in $(29a)_1$. The integrations in the x-direction (products of two trigonometric functions of x) are performed analytically, while the integrations in the y-direction (polynomials in y of up to 12th degree) are evaluated numerically by means of Gauss' integration formula. The results of the x-integrations show that $a_{\iota\kappa\mu\nu} = 0$ when the sum of the wave numbers associated with the four modes $(m_\iota + m_\kappa + m_\mu + m_\nu)$ is an uneven number.

Having determined approximate values of the coefficients $a_{\iota\kappa\mu}$ and $a_{\iota\kappa\mu\nu}$ by means of the finite strip method, we then insert these coefficients in the nonlinear equilibrium equations (15). The numerical solution of these nonlinear equations is determined by Newton-Raphson iteration.

Dependence of $\mathbf{u}_{\iota\kappa}$ *and* $a_{\iota\kappa\mu\nu}$ *on load factor* λ

The presence of the term $\lambda\dot{\Phi}_{11}$ in (13a) shows that the 2nd order displacements $\mathbf{u}_{\iota\kappa}$ will depend on the load factor λ, and it follows from (16a) that the four-index coefficients $a_{\iota\kappa\mu\nu}$ will likewise depend on λ. In order to gain insight into this dependence, the following results will often prove useful: Consider a discrete system with M degrees of freedom (e.g. the finite strip model of the thin-walled beam). Such a system possesses M linearly independent and mutually orthogonal eigenvectors. Suppose that we choose N_b buckling modes $\mathbf{u}_\iota$ ($\iota = 1, 2, \ldots, N_b$). Then the residual displacement $\mathbf{v}$, see (10), which is orthogonal to the buckling modes $\mathbf{u}_\iota$, can be expressed as a linear combination of the remaining eigenvectors $\mathbf{u}_\Gamma$, i.e.

$$\mathbf{v} = \sum_{\Gamma = N_b+1}^{M} \mathbf{u}_\Gamma\,\xi_\Gamma \tag{31}$$

(note that capital Greek indices have the range $N_b + 1$ to M). Using (13a) and (7a), it can then be shown (see Goltermann, 1985, Appendix 2) that the 2nd order displacement and the 4-index coefficients can be represented in the form

$$\mathbf{u}_{ij} = \sum_{\Gamma} \frac{a_{ij\Gamma}}{(\lambda - \lambda_{\Gamma})} \mathbf{u}_{\Gamma} \tag{32a}$$

$$a_{ijkl} = \tfrac{1}{6}\Phi_{1111}[\mathbf{u}_i, \mathbf{u}_j, \mathbf{u}_k, \mathbf{u}_l] + \frac{2}{3}\sum_{\Gamma} \frac{a_{ij\Gamma}a_{kl\Gamma} + a_{ik\Gamma}a_{jl\Gamma} + a_{il\Gamma}a_{jk\Gamma}}{(\lambda - \lambda_{\Gamma})} \tag{32b}$$

where $a_{ij\Gamma}$ is given by formulae (27) and (28) with $\mathbf{u}_k$ replaced by $\mathbf{u}_{\Gamma}$, and it is assumed that the additional eigenvectors $\mathbf{u}_{\Gamma}$ are normalized according to (9a).

It can be seen from (32a) and (32b) that $\mathbf{u}_{ij}$ and a_{ijkl} will, in general, possess singularities for $\lambda = \lambda_{\Gamma}$. This observation can be used to explain the reason for our previous remark (Section 3) about the necessity of including among the buckling modes all the modes with eigenvalues in the interval of load factors $\lambda_a \leq \lambda \leq \lambda_b$ with which we are concerned in the interaction analysis. It will now be seen that, when these modes are included among our buckling modes, they will not appear in the summation in (32a). This means that we in fact suppress the singularities in the said interval, and the resulting 2nd order displacements $\mathbf{u}_{ij}$ and 4-index coefficients a_{ijkl} will therefore be continuous functions of λ in the interval $\lambda_a \leq \lambda \leq \lambda_b$.

However, for the types of thin-walled beams with which we are concerned, it is found that there exists a whole cluster of closely spaced eigenvalues in an interval just above the smallest eigenvalue of the local modes. It follows from our previous remarks that if we wish to use the perturbation method for λ-values in this interval, we must then include among our buckling modes all the local modes associated with these closely spaced eigenvalues. Although it would be possible, in theory, to perform such a multi-mode analysis, it is not a practical proposition, as it would increase the amount of calculations by an almost prohibitive amount. However, it will be shown in Part II of our paper (Goltermann and Møllman, 1989) that, for certain types of local imperfections, we may in fact omit the multi-mode analysis and restrict ourselves to a 3-mode analysis, and still obtain sufficiently accurate results. The calculations in Part II will therefore mainly be restricted to 3-mode analyses (or in a few cases 2-mode analyses) of beams with doubly symmetric cross-sections, in which the buckling modes comprise one global mode $\mathbf{u}_1$ together with one or two local modes, $\mathbf{u}_2$ and $\mathbf{u}_3$. Since we use only three (or two) buckling modes, the singularities in the 4-index coefficients have not been suppressed (this would require additional local buckling modes), and some of the coefficients in the nonlinear equilibrium equations (15) of the 3-mode analysis may therefore depend strongly on λ.

It will be shown in Part II that the 2nd order displacements $\mathbf{u}_{11}$, $\mathbf{u}_{22}$, and $\mathbf{u}_{33}$, and the associated 4-index coefficients a_{1111}, a_{2222}, and a_{3333}, are only slightly influenced by the value of λ. On the other hand, the mixed 2nd order displacements $\mathbf{u}_{12}$ and $\mathbf{u}_{13}$ (associated with the global mode and one local mode), and corresponding mixed 4-index coefficients such as a_{1122} and a_{1133}, may depend strongly on λ. However, it is shown in Part II that, for the type of local imperfections considered in the paper, the λ-sensitive coefficients do not appear in our equations, and we obtain a valid solution by using a 3-mode analysis.

5. CONCLUDING REMARKS

A method has been developed for the analysis of nonlinear mode interaction in thin-walled beams. The method is based on the finite strip method in conjunction with Koiter's asymptotic theory of stability, and it is capable of describing the whole range of behaviour of these structures from local to global buckling. A subsequent paper by the authors describes some applications of the method to thin-walled beams having doubly symmetrical cross-section and shows that significant mode interaction and imperfection sensitivity occur for these structures. Further discussion of the performance and characteristics of the method will be deferred to our second paper, where the results of the numerical examples are at our disposal.

REFERENCES

Benito, R. (1983). Static and dynamic interactive buckling of plate assemblies. Doctoral thesis, Washington University, Saint Louis.

Benito, R. and Sridharan, S. (1985a). Interactive buckling analysis with finite strips. *Int. J. Numer. Meth. Engng* **21**, 145–161.

Benito, R. and Sridharan, S. (1985b). Mode interaction in thin-walled structural numbers. *J. struct. Mech.* **12**, 517–542.

Bradford, M. A. and Hancock, G. J. (1984). Elastic interaction of local and lateral buckling of beams. *Thin-walled Struct.* **2**, 1–25.

Byskov, E. (1983). An asymptotic expansion applied to van der Neut's column. In *Collapse: the Buckling of Structures in Theory and Practise* (Edited by J. M. T. Thompson and G. W. Hunt), pp. 269–281. Cambridge University Press, Cambridge.

Byskov, E. and Hutchinson, J. W. (1977). Mode interaction in axially stiffened cylindrical shells. *AIAA Jl* **15**, 941–948.

Goltermann, P. (1985). Interaction between local and global stability in thin-walled beams (in Danish). Dept. of Struct. Engng Tech. Univ. of Denmark, report No. 207.

Goltermann, P. and Mollmann (1989). Interactive buckling in thin-walled beams—II. Applications. *Int. J. Solids Structures* **25**, 729–749.

Graves-Smith, T. R. (1969). The ultimate strength of locally buckled columns of arbitrary length. In *Thin Walled Steel Structures* (Edited by K. C. Rockey and H. V. Hill), pp. 35–60. Crosby, Lockwood & Son, London.

Graves-Smith, T. R. and Sridharan, S. (1978). A finite strip method for the post-locally buckled analysis of plate structures. *Int. J. Mech. Sci.* **20**, 833–842.

Graves-Smith, T. R. and Sridharan, S. (1980a). The local collapse of elastic thin-walled columns. *J. struct. Mech.* **8**, 471–489.

Graves-Smith, T. R. and Sridharan, S. (1980b). Elastic collapse of thin-walled columns. In *Thin-Walled Structures* (Edited by J. Rhodes and A. C. Walker), pp. 718–730. Granada, St. Albans.

Hancock, G. J. (1981). Interaction buckling in I-section columns. *J. struct. Div., ASCE* **107** (ST1), 165–179.

Hancock, G. J. (1981). Nonlinear analysis of thin sections in compression. *J. struct. Div., ASCE* **107** (ST 3), 455–471.

Koiter, W. T. (1945). On the stability of elastic equilibrium, Thesis. H. J. Paris, Amsterdam (in Dutch). English translation NASA, Tech. Trans. F-10, 833 (1967).

Koiter, W. T. (1976). A general theory of mode interaction in stiffened plate and shell structures. Lab. of Engng Mech., Delft, report WTHD No. 91.

Koiter, W. T. (1980). Basic principles of elastic stability. In *Thin Shell Theory, New Trends and Applications* (Edited by W. Olszak), pp. 65–75, Springer, Wien.

Koiter, W. T. (1982). The application of the initial post-buckling analysis to shells. In *Buckling of Shells* (Edited by E. Ramm), pp. 1.1–1.15. Inst. für Baustatik, Univ. Stuttgart.

Koiter, W. T. and Kuiken, G. D. C. (1971). The interaction between local buckling and overall buckling on the behaviour of built-up columns. Lab. of Engng Mech., Delft, report WTHD No. 23.

Koiter, W. T. and Pignataro, M. (1976). A general theory for the interaction between local and overall buckling of stiffened panels. Lab. of Engng Mech., Delft, report WTHD No. 83.

Mollmann, H. (1984). Buckling with mode interaction: the complete perturbation expansion. Dept. of Struct. Engng, Tech. Univ. of Denmark, internal report.

Neut, A. van der (1969). The interaction of local buckling and column failure of thin-walled compression members. In *Proc. 12th Int. Congr. Appl. Mech.* (Edited by M. Hetenyi and W. G. Vincenti), pp. 389–399. Springer, Berlin.

Neut, A. van der (1976). Mode interaction in stiffened panels. In *Buckling of Structures* (Edited by B. Budiansky), pp. 117–132. Springer, Berlin.

Pedersen, C. (1982). Stability properties and non-linear behaviour of thin-walled elastic beams of open cross-section, Parts 1 and 2. Dept. of Struct. Engng, Tech. Univ. of Denmark, reports Nos. 149 and 150.

Sridharan, S. (1983). Doubly symmetric interactive buckling of plate structures. *Int. J. Solids Structures* **19**, 625–641.

Sridharan, S. and Ashraf Ali, M. (1985). Interactive buckling in thin-walled beam-columns. *J. Engng Mech. Div. ASCE* **111**, 1470–1486.

Sridharan, S. and Ashraf Ali, M. (1986). An improved interactive buckling analysis of thin-walled columns having doubly symmetric sections. *Int. J. Solids Structures* **22**, 429–443.

Sridharan, S. and Graves-Smith, T. R. (1981). Postbuckling analysis with finite strips. *J. Engng Mech. Div. ASCE* **107** (EM5), 869–888.

Svensson, S. E. (1976). Ultimate strength of the longitudinally stiffened, compressed plate. *Bygningsstatiske Meddelelser* **47**, 1–20.

Svensson, S. E. and Croll, J. G. A. (1975). Interaction between local and overall buckling. *Int. J. Mech. Sci.* **17**, 307–321.

Tvergaard, V. (1973). The imperfection sensitivity of a wide integrally stiffened panel under compression. *Int. J. Solids Structures* **9**, 177–192.